UUITP-7/95
FIAN/TD-19/95
ITEP/TH-9/95
revised version

Towards effective topological gauge theories on spectral curves

A.Gorsky [1]

ITEP, Moscow, 117 259, Russia

A.Marshakov [2]

Theory Department, P. N. Lebedev Physics Institute , Leninsky prospect, 53, Moscow, 117924, Russia
and
ITEP, Moscow 117259, Russia

Abstract

We discuss a general approach to the nonperturbative treatment of quantum field theories based on existence of effective gauge theory on auxiliary "spectral" Riemann curve. We propose an effective formulation for the exact solutions to some examples of $2d$ string models and $4d$ supersymmetric Yang-Mills theories and consider their natural generalizations.

[1] E-mail address: gorsky@vitep3.itep.ru
[2] E-mail address: mars@lpi.ac.ru

Recently the development of matrix model technique has lead to understanding the role of integrable structures in string theory [1, 2, 3, 4]. One of the ways to interpret this fact is to consider appearing integrable equations as equations of motion in (hypothetic) string field theory and it looks natural to search for a field-theoretical formulation of this target-space theory directly, i.e. without any referring to $2d$ world-sheet. Moreover there exists a direct analog of this phenomenon for the $4d$ theory [5, 6] where four-dimensional space-time plays the role of the world sheet and the exact solution can be given in similiar terms. In both cases the most adequate formulation is given in terms of the integrable systems of "KP/Toda type".

In this letter we will argue that the appearence of integrable systems can be treated within some unified framework and the nonperturbative information can be read off from the topological effective gauge theories. Usually one starts with a bare theory defined on a "world-sheet" and considers Toda-type equations coming commonly from the symmetries of a bare theory. Next, using the symmetries of the bare model one can specify the solution which corresponds directly to the nonperturbative regime and depends on finite number of moduli parameters of the theory (at least we will restrict ourselves for such cases). Finally, we will use the fact that such solutions can be connected with topological gauge theories where the auxiliary (from the world-sheet point of view) spectral curve appears as a (part of) target-space. More strictly the spectral curve of arising integrable system of Hitchin type [8] is a cover of a surface where the auxiliary gauge theory is defined. One may hope that this is a generic scheme valid for a wide class of the nonperturbative string and gauge field theories, below we will demonstrate the main features of this scheme considering existing up to now $2d$ and $4d$ examples.

First we will consider the simplest example when the role of the world-sheet theory is played by a discrete matrix model and show that it can be reinterpreted in terms of a generalized $2d$ YM model. Then we will discuss a relation between the (perturbed) G/G gauged WZW model and some class of $2d$ topological theories. Finally, we propose the effective theory for $4d$ $N = 2$ SUSY YM and its generalizations in terms of a so called holomorphic generalized YM model on torus. The quantum characteristics of the effective theories like spectrum and wave functions and their relevance to the nonperturbative characteristics of bare models are also discussed.

1. Let us turn directly to the simplest example and consider a *discrete* matrix model, where instead of integration over the functions on world sheet $f(z, \bar{z})$ we are summing over matrices or functions H_{ij} of two discrete variables. The role of the reparametrizations of the world-sheet is played by a sort of "gauge" symmetry $H \to UHU^{-1}$ or in the infinitesimal form $\delta H = [\epsilon, H]$ which is similiar to (complex) one-dimensional world-sheet symmetries generated by $\delta_\epsilon H = \partial_z^{|i-j|} H + ...$ Since the gauge symmetry corresponds to reparameterizations the moduli space of gauge connections is direct analog of moduli space of complex structures in string theory.

The main observation of this part is that for discrete matrix model an effective target-space formulaton is given by a degenerate case of the generalized two-dimensional YM theory. We should point out that in general the effective theory is defined on a spectral curve though it is not seen here explicitely since we have a degenerate case of a rational curve. This is actually one of very important questions and we will return to this discussion in detail below. Here the role of a spectral curve is played by a circumference $0 < x < 2\pi$ where one can write

the generalized [1] $2d$ YM action

$$S = \mathrm{Tr}\int_{dxdt}(F(A)\Phi - V(\Phi) + A_i J_i) = \\ = \mathrm{Tr}\int_{dxdt}(\Phi\partial_t A_x - V(\Phi) + A_t(\partial_x\Phi + [A_x,\Phi] - I)) \tag{1}$$

giving the equations of motion

$$\partial_t\Phi + [A_t,\Phi] = J_x = 0 \tag{2}$$

$$F_{tx}(A) = [\partial_t + A_t, \partial_x + A_x] = \partial_t A_x - \partial_x A_t + [A_t, A_x] = V'(\Phi) \tag{3}$$

and the Gauss law

$$\partial_x\Phi + [A_x,\Phi] = J_t = I \tag{4}$$

The Hamiltonian is

$$H = \mathrm{Tr}\int_{dx}(V(\Phi) + A_t(\partial_x\Phi + [A_x,\Phi] - I)) \tag{5}$$

and it is possible to define additional (commuting) flows

$$\frac{\partial\Phi}{\partial t_k} = \{H_k,\Phi\} \qquad H_k = \mathrm{Tr}\Phi^k \tag{6}$$

where the Poisson bracket is induced by $\Omega_{YM} = \mathrm{Tr}\int_{dx}\delta\Phi\wedge\delta A_x$. In terms of integrable systems eq. (4) is a reduction constraint, so that Φ determined as a solution to (4) becomes the Lax operator of a discrete Calogero system. Then (2), (6) are the Lax evolution equations in the corresponding finite-dimensional integrable system. Finally, (3) is vacuum equation, or the initial condition for the dynamical system (2), (6).

Now the relation to discrete matrix model appears when one takes t-independent solutions so that (2) and (3) reduce to

$$[A_t,\Phi] = 0 \\ -\partial_x A_t + [A_t, A_x] = V'(\Phi) = \sum k t_k \Phi^{k-1} \tag{7}$$

The first one tells that A_t is some function of Φ [2] while the second says that the minima of the Hamiltonian (8) $V'(\Phi) = 0$ correspond to the solutions to $[D_x, A_t] = [\partial_x - A_x, A_t] = 0$ so that A_t commutes with the "shift" operator D_x. Thus, the first term in the action (1), vanishes and one gets the action on the vacuum solution

$$S_{vac} = \mathrm{Tr}V(\Phi) \tag{8}$$

Now one can easily notice that (8) looks similiar to the effective action of discrete 1-matrix model [13]. To clarify the similiarity let us after the redefinition of the variables

$$q_i = \phi_i + (i-1)\Delta \quad g = g_0 e^{\frac{\Delta}{2}} \quad \Delta\to\infty \tag{9}$$

[1] which transforms to the ordinary $2d$ YM action after integrating over Φ for $V(\Phi) = \Phi^2$. The stringy interpretation of the pure $2d$ YM theory was proposed in [7].

[2] actually A_t is a degeneration of the following expression in the elliptic case

$$A_{t,ij} = \delta_{ij}\left(-\wp(\lambda) + 2\sum_{k\neq i}\wp(x_i - x_k)\right) + 2(1-\delta_{ij})\Phi'(x_i - x_j;\lambda)$$

take the limit [9] from Calogero to nonperiodic Toda chain

$$
\begin{gathered}
H = \sum \frac{1}{2} p_j^2 + g^2 \sum_{i>j} \wp(q_i - q_j) \rightarrow \\
\rightarrow \sum \frac{1}{2} p_j^2 + g^2 \sum_{i>j} \sinh^{-2}(q_i - q_j) + const \rightarrow \sum \frac{1}{2} p_j^2 + g_0^2 \sum e^{\phi_j - \phi_{j+1}}
\end{gathered} \tag{10}
$$

Keeping $g_0 \to \infty$ means that the Toda limit with $g^2 = \nu(\nu - 1)$ implies $\nu \to \infty$. The Toda chain τ-function

$$
\frac{\tau_N}{\tau_0} = \det C_{N\times N} = e^{\sum_{i=1}^{N} \phi_i} = \frac{\tau_N(t)}{\tau_0(t)} \sim e^{\mathrm{Tr} \int a_x dx} \sim \det \exp e^{\int a_x dx} \tag{11}
$$

is the determinant of the moment matrix (see for example [10]), [11] [3] which in this example is the monodromy matrix of the YM field $a_x = diag(\phi_1 ... \phi_N)$. Now, string equation

$$
\sum_{k>0} kt_j \frac{\partial}{\partial t_{k-1}} \tau_N(t) = \sum kt_k \mathcal{H}^{k-1} = \sum_{k>0} kt_k \mathrm{Tr} L^{k-1} = 0 \tag{12}
$$

can be interpreted as an equation for the Lax operator $L[\phi(t)]$ and the matrix model *effective* action [13]

$$
S = \mathrm{Tr} \sum_k t_k L^k + S_0 + \frac{1}{2} \sum_n \phi_n = S_{vac} + \frac{1}{2} \mathrm{Tr} A_x \tag{13}
$$

almost coincides with (8) after Toda reduction. The additional term $\mathrm{Tr} A_x$ [4] is related to more tiny effects which will be discussed later. This actually demonstrates the necessity of quantization of the effective theory.

Consider now the solutions to string equation in the simplest cases. From (12) it follows for $t_k = 0$ for $k \geq 2$ we have

$$
\sum_i t_2 \frac{\partial \phi_i}{\partial t_1} \sim \sum p_i = N t_1 \tag{14}
$$

having the sense of total momentum "flow" in t_1-direction for zero higher times [5]. String equation results in a peculiar Δ-dependence for the YM coupling constant. Using relation [3]

$$
n = \sum_{k>0} kt_k \frac{\langle n-1 | \lambda^{k-1} | n \rangle}{\langle n | n \rangle} e^{\phi_n - \phi_{n-1}} \tag{15}
$$

giving for only $t_2 \neq 0$

$$
\begin{gathered}
e^{\phi_n - \phi_{n-1}} = \frac{n}{2t_2} \\
\phi_n = \log n! - n \log 2t_2
\end{gathered} \tag{16}
$$

[3] being for the 1-matrix model

$$
C_{ij} = \int d\lambda e^{-V(\lambda)} \lambda^{i+j-2}
$$

[4] Coming from

$$
\sum_{k>0} kt_k \langle n | \lambda^{k-1} | n \rangle = \sum_{k>0} kt_k \left(L^{k-1} \right)_{nn} = 0
$$

and

$$
\left(\frac{\partial V}{\partial L} \right)_{n-1,n} = \sum_{k>0} kt_k \left(L^{k-1} \right)_{n-1,n} = \frac{n}{L_{n,n-1}} = n e^{-\frac{1}{2}(\phi_n - \phi_{n-1})}
$$

[5] For generic point in the space of times (or coupling constants) one has instead of (15) the condition

$$
\sum kt_k H^{k-1} = t_1 N + 2t_2 P + 3t_3 E + ... = 0
$$

which for example for nonzero t_3 and zero higher times has a sense of "energy" dissipation.

one can single out the linear piece and compare with (9). The result gives

$$t_2 = g^2(\Delta) = e^{\Delta} \tag{17}$$

Now we will discuss another close example concerning the effective description in terms of the gauged G/G [6] Wess-Zumino-Witten model which is a direct generalization of $2d$ YM system [12]. The Lagrangian defined on a ("spectral") surface has the following form

$$\begin{aligned} S = kS_{WZW}(g) + \frac{ik}{2\pi}\mathrm{Tr}\int(A_z g^{-1}\bar\partial g + g\partial g^{-1}A_{\bar z} + gA_z g^{-1}A_{\bar z} - A_z A_{\bar z}) \\ +V(g) + \nu\omega_{\bar z}A_t J_z \end{aligned} \tag{18}$$

with the Hamiltonian

$$V(g) = \frac{1}{2}\sum(t_k \mathrm{Tr} g^k + \bar t_k \mathrm{Tr} g^{-k}) \tag{19}$$

The equations of motion are

$$\begin{aligned} g^{-1}\bar\partial g + g^{-1}A_{\bar z}g - A_{\bar z} = J_{\bar z} \\ F_{z,\bar z}\left[A_z, g^{-1}\partial g + g^{-1}A_{\bar z}g\right] \equiv \bar\partial A_z - \partial(g^{-1}\bar\partial g + g^{-1}A_z g) \\ + \left[A_z, (g^{-1}\bar\partial g + g^{-1}A_z g)\right) = g^{-1}V'(g) \end{aligned} \tag{20}$$

and generalized Gauss law is

$$g\partial g^{-1} + gA_z g^{-1} - A_z = J_z \tag{21}$$

The observables in the theory are $\mathrm{Tr}_V g$ or the Wilson loops in different representations.

Now, let us remind how the Ruijsenaars dynamics appears in the G/G theory [12]. Again, one should consider the special sources $J_{ij} = 1_{ij} - \delta_{ij}$ and after the resolution of (21) it appears that in the gauge $A = diag(q_1, \ldots, q_n)$ g becomes the Lax operator for the trigonometric (or hyperbolic) Ruijsenaars system

$$L_{ik} = e^{p_i}\prod_{i\neq j}\frac{\sin[(\frac{\pi}{k})(q_i - q_j + \frac{1}{k}]}{\sin[(\frac{\pi}{k})(q_i - q_j]} \tag{22}$$

where q_i and p_i are canonically conjugated variables.

On its own the Ruijsenaars system is related with the pole solution to the 2D Toda lattice [15]. Consider the elliptic solutions of the Toda lattice in (discrete) x variable, namely let us assume that $\phi(x+\eta, t, \bar t) - \phi(x, t, \bar t)$ is an elliptic function of the variable x. Then ϕ is a solution of the Toda equations if and only if the poles of the solutions x_i

$$e^{\phi} = \prod_{i=1}^{N}\frac{\sigma(x - x_i + \eta)}{\sigma(x - x_i)} \tag{23}$$

move with respect to the linear combination of the time variables t, $\bar t$ according to the Hamiltonian equations of motion with $H = \mathrm{Tr}L + \mathrm{Tr}L^{-1}$ where L is the Lax operator for the elliptic Ruijsenaars system. Here we are interested in the trigonometric degeneration of this relation. Note that comparing (22) with (23) we see that $\eta = \frac{1}{k}$.

Now one can remind that [16] that GWZW can be considered as a low energy limit of the $2d$ SUSY topological σ-model with $U(N)$ gauge symmetry and kN chiral matter multiplets, so that if one integrates out the chiral matter the resulting low energy effective action is given by GWZW theory.

[6]For example $G = SU(N)_k$ or $U(N)_{k-N,k} \equiv SU(N)_{k-N}\otimes U(1)_k$

Let us turn now to the integrable structure behind the topological σ models, where [17] the underlying integrable system is again 2D Toda lattice. The main input for the integrable structure (see for example [18] and references therein) is the (perturbed) quantum cohomology ring generated for $CP(N)$ by a factorization over

$$x^n = \beta \tag{24}$$

where β is the action calculated on an instanton. The corresponding coupling constants play the role of the "times" in the integrable hierarchy, they contain the coupling to the Kähler class $\log\beta = \frac{i}{g^2} + \theta$ where g is the bare coupling constant of the σ model and θ is bare θ-term.

The Toda lattice variables

$$\phi_i = lng_{i\bar{j}} - \frac{2i - N + 1}{2N} ln|\beta|^2 \tag{25}$$

are the eigenvalues of the ground state metric [17] related with the nonabelian Berry connection and can be extracted from the two-point topological correlators. The relation (25) reminds a lot (16) from the previous example.

It is necessary to mention that unlike [17], where reduction to the radial coordinate $|\beta|$ with the Painleve-3 solution arose due to the equal number of instantons and anti-instantons, we do not impose such a condition and obtain another solution to the Toda lattice equation. If one tends the level of the theory to infinity there will be the topological YM theory on one side and the continious KP equation on the other.

2. It is known that nonlinear σ models play the role of the playground for the nonperturbative approaches to $4d$ QCD. Recently [5] the nonperturbative solution to $N = 2$ SYM theory has been derived which provides exact mass spectrum and the low-energy effective action. It was shown in [6] [7] that there exists an integrable structure behind this solution so that the massive spectrum is determined by the periods and effective action is a logariphm of the τ-function of the Whitham hierarchy.

The main step of the solution is the determination of the spectral curve with some additional structure which carries all nonperturbative information about the theory. The spectral curve plays the role of the initial data (boundary condition) and we are going to define an effective topological theory on this curve. Having some picture for the $2d$ theory one can think that in $4d$ theory the variables t, $\bar{t}$, x are also the same combinations like $e^{\frac{i}{g^2}+\theta}$, $e^{\frac{i}{g^2}-\theta}$ and the Chern number, but the accurate analysis needs of course the determination of the topological subsector. We will not discuss this important point here, however as it was mentioned in [6] the desired $SU(N)$ curve found in [23] arises in a periodic Toda-chain system.

In what follows we are going to consider the periodic Toda chain as a degenerate case of a certain Hitchin-type [8] dynamical system on target-space torus and related with the affine $sl(N, \mathbf{C})$ algebra. We will try to demonstrate that these sort of formal generalizations are also quite interesting from physical point of view.

Let us start with the *holomorphic* YM theory action define on a torus

$$S = \mathrm{Tr} \int_{d^2zdt} \omega_{\bar{z}} F_{tz}(A)\Phi - \omega_z V(\Phi) + \nu A_t J_{\bar{z}} \omega_{\bar{z}} \tag{26}$$

[7] and this idea was developed in [19], [20], [21], [22]

where ω_z is the holomorphic 1-differential, and equations of motion generalize (2), (3), (4) and (20)

$$\partial_t \Phi + [A_t, \Phi] = 0$$
$$F_{tz}(A) = [\partial_t + A_t, \partial_z + A_z] = \partial_t A_z - \partial_z A_t + [A_t, A_z] = V'(\Phi) \tag{27}$$

$$\partial_z \Phi + [A_z, \Phi] = J_z \tag{28}$$

The Hamiltonian is now $H = \mathrm{Tr} \int d^2 z Q(\Phi)$ where $Q(\Phi)$ is an invariant polynomial on the Lie algebra $sl(N, C)$, and finite-dimensional dynamical system appears as before after the resolution of the Gauss law constraint so that [24] Φ occures to be a Lax operator of an elliptic Calogero-Moser system or a particular case of the Hitchin system on torus. Now the key point is that the periodic Toda chain arises as a limit of the elliptic Calogero system and hence of the corresponding effective topological field theory on target space.

Starting from the initial elliptic Calogero (9) we have to proceed now in a slightly more delicate way than before. One has to introduce the new Toda variables by $q_i = \phi_i + (i-1)\tau$ and put [9]

$$g^2 = e^{\tau} = e^{\frac{\tilde{\tau}}{N}} \qquad \tilde{\tau} \to \infty \tag{29}$$

As a result we obtain the periodic Toda chain when the Lax operator of elliptic Calogero

$$L_{Cal,ik}(z) = \delta_{ik} p_k + i(1 - \delta_{ik}) \frac{\sigma(q_i - q_k + z)}{\sigma(z)\sigma(q_i - q_k)} \tag{30}$$

reduced to

$$L_{Toda,jk}(w) = \delta_{jk} p_k + \delta_{j,k+1} \exp(q_j - q_{j-1}) - (i)^N w \delta_{jN} \delta_{k1}$$
$$-i^{-N} w^{-1} \delta_{j1} \delta_{kN} \exp(q_1 - q_N) \tag{31}$$

The Calogero spectral curve which is in this case the N-sheet covering of the torus defined by

$$\det_{ij} \left(L_{Cal}(z) - \lambda \right) = 0 \tag{32}$$

degenerates to

$$w + \frac{1}{w} = 2 P_{Toda}(\lambda) \qquad y^2 = P_{Toda}^2 - 1 \tag{33}$$

where $2y = w - w^{-1}$. Note that if we keep the coupling constant in Toda after the short calculation one gets Λ^N instead of unity in the last term in (33). This clearly indicates that this term plays the role of the instanton correction just as in $2d$ case.

Thus it turns out that the Seiberg-Witten (and their $SU(N)$ generalization) curves are special limits of the N-sheet covering of the torus. Moreover there exists a certain generalization [8] again having the form of (32) where $L_{Cal}(z)$ now should be replaced by the Lax matrix of the Ruijsenaars system

$$L_{ellR-s,jk}(z) = \exp(\beta p_j) V_j(q) \frac{\sigma(q_j - q_k + z)\sigma(\eta)}{\sigma(z)\sigma(q_j - q_k + \eta)} \qquad V_j = \prod_{j \neq k} f(q_j - q_k)$$
$$f^2_{ellR-s}(x) = \sigma^2(\eta) \left(\wp(\eta) - \wp(x) \right) \qquad f^2_{rel-Toda}(x) = 1 + \tau^2 \beta^2 e^x \tag{34}$$

where the last expression corresponds to the degenerate periodic relativistic Toda case [25]. We conjecture that generic Calogero - Ruijsenaars type curve might correspond to some $4d$ gauge systems coupled to special

[8] related hopefully to the quantum affine algebras

matter. The case with the fixed flavour number and arbitrary mass scale of matter, being a special case of (30) is discussed in [21], and one might expect the appearence of more general curves of the (32) type for generic N_f. Typically the level of Kac-Moody algebra is related with the number of the fermionic flavours so we can conjecture that such curves appear if one adds the $k = N_f$ flavour matter to the theory on the world-sheet. It would be very important to get the proper world sheet theory because it would provide some view on a role of the quantum group and Sklyanin algebra in essentially $4d$ theories.

The generalization from Toda to Calogero case can be also thought of as follows. One can think that the papameter τ serves as a measure of the "vacuum transition amplitudes" and the reduction to Toda system means that the amplitude is "small" and only the transitions between the neighbour vacua survive. The transition from Calogero to Toda looks like the transition from the "liquid" phase to the "crystal" one.

Note that there are other possible generalizations of the spectral curve. One can add the internal degrees of freedom to the "particles" – that would lead to the "nonabelian" integrable systems similiar to those of [15], consider theory on higher genus curves (see for example [26]) or add additional marked points. We conjecture that all these possibilities can be realized if one starts with more complicated bare theory, i.e. takes more initial coupling constants or consider more general symmetry breaking scheme.

3.Finally, we will try to discuss the meaning of quantization of arising effective theory. First let us mention that in $2d$ case from string theory point of view this is a rather natural procedure which should lead to the second-quantized string field theory in its original sense. In the $4d$ situation this is not as clear but we would still argue that the quantum characteristics of the effective topological gauge theory (wave functions, spectrum etc) are relevant for the description of bare model.

Let us start again from the most simple example of discrete matrix model. As we have tried to argue above the effective theory can be represented in the form of generalized YM model which reduces on particular solutions to the effective discrete matrix model action written in terms of the Lax operator. Let us notice first that the wave function of the generalized YM theory in the limit $\nu \to 0$ acquires the form of the GKM partition function [12, 27] giving for this case the generating function for the exact correlators in $2d$ topological gravity models. This analogy as well as relation of the Ruijsenaars wave functions (or the Harish-Chandra functions) to the soliton S-matrices [31] allows one to hope that the wave functions of the systems considered above can be treated as generating functions for nonperturbative correlators. These wave functions are known to be certain one-point conformal blocks on torus [32],

$$\Psi_\lambda(\mathbf{t}, q) = \frac{\mathrm{Tr}_\lambda \left(V q^{L_0} e^{\mathbf{tH}}\right)}{\mathrm{Tr}_{-\rho} \left(V q^{L_0} e^{\mathbf{tH}}\right)} \tag{35}$$

being the solutions to the Knizhnik-Zamolodchikov-Bernard equations in the $\widehat{sl(N, \mathbf{C})}$ WZW theory at critical level $K = -C_V$. The limits (9), (29) imply that one has to put $C_\lambda = e^\Delta$ for $\Delta \to \infty$ which means that for the Toda theory we end up with the infinite-dimensional representation. In such limit the wave function (35) turns into the periodic Toda-chain wave function which should be related with the generating function for the nonperturbative $4d$ theory.

Another ($\nu \to \infty$) limit giving rise to the discrete matrix model itself can be reformulated in terms of the following Lax operator $\Phi(A_x) \to L(\phi) \to W(X) = X + \frac{n}{X}$ playing the role of LG superpotential in GKM-type

theories. In this formulation the corresponding action [29] $V(X) = \int dXW(X) = \frac{1}{2}X^2 + n\log X$ naturally separates into quadratic classical "YM" piece and the logariphmic one-loop correction and serves as a potential in the exact partition function [11]. Similiar superpotentials arise in the case of topological σ-models [17].

One could consider here two naively different types of quantization: the first deals with the Whitham type deformation of the spectral curve [30], another quantizes the gauge theory on the undeformed surface. The first one can be identified with the renormalization group flow [6] in world sheet theory and in $2d$ case leads to the quantization of finite-gap potentials [33] and to the GKM-type theory. From the point of view of gauge theory it leads to the appearence of Grassmannian-like fermionic constructions when effective action arises as a result of the integration over fermionic degrees of freedom on spectral curve.

The second way should have a meaning of the derivation of a kind of "effective action" for world sheet theory before renormalization. For example in the $SL(2)$ case the equations of motion

$$\frac{d^2\phi}{dt^2} + g\sin\phi = 0 \qquad H = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - g\cos\phi \tag{36}$$

can be easily integrated giving

$$t = \frac{1}{\sqrt{2g}}\int \frac{dz}{\sqrt{(z-u)(z^2-1)}} \tag{37}$$

with $u \equiv \frac{H}{g}$ thus giving rise to the $N = 2$ SUSY YM spectral curve [5, 6]. This theory should be considered as an effective one coming from integration over the (Grassmannian) fermions so that the full spectrum can be considered as a sum of the "fermionic one" where the fermions are solutions of auxiliary linear problem and the perturbations of the above classical solution. The first part gives rise to the Seiberg-Witten terms of the type $E_{ferm} \sim \int dS = \int Edp$ and one can hope that the spectrum of auxiliary theory itself might be related with the spectrum of "extra states" of original theory similiar to pomerons.The whole spectrum of exitations will be considered elswhere.

Let us finally discuss the quantization from more general point of view. As usual we deal with the world sheet theory which after integration over the world-sheet variables gives us some effective target-space model where we interest in the low-frequency "visible" part of the spectrum. The string equation gives us a renormalization group equation while a particular critical point corresponds to some β-function fixed point condition. Now, at any concrete fixed point we end up with the *finite-dimensional* subspace in the space of the coupling constants, i.e. in other words each fixed point correspond to a renormalizable theory in the common field-theoretical language (one should not add any additional counter-terms to renormalize the theory). The string equation considered in the vicinity of such critical point acquires the form of the characteristic solution to particular Whitham equations. The main subtle point here is that the finite-dimensional Whitham integrable system is singled out by some natural requirement of periodicity and it is natural to ask what is the sense of the periodicity condition in field theory. The only possible candidate one has in mind is the θ-parameter (or its conjugate K).

In this letter we have proposed the approach when the integrable structure corresponding to the nonperturbative solutions of some $2d$ and/or $4d$ world - sheet theory is encoded in the effective topological gauge theory on some "spectral" curve. This effective topological theory governs the dynamics on the moduli space of bare model which together with the spectral curve describes the solution. We have conjectured that the quantization

of the effective theory is not meaningless from the point of view of the original problem hence the reformulation of the problem in canonical field-theoretical language is attractive from many different points of view.

The reason for the appearence of an effective gauge theory can be commented in the following way. If there exists a degenerate vacuum state in the bare theory one can define a (non Abelian) Berry connection. It is this connection which can be considered as a connection on "spectral curve" discussed above. The presence of marked points on spectral curve (= "dispersion law") is related to the level crossing and corresponds to the vanishing of mass gap for quasiparticles providing some symmetry breaking mechanism in bare theory.

Of course, our understanding of many points discussed above is far from being complete and we are going to return to these problems elsewhere.

We are indebted to I.Krichever, A.Mironov, A.Morozov, N.Nekrasov and V.Rubtsov for useful discussions. We are grateful to J.Ambjorn for the hospitality in the Niels Bohr Institute were this work was initiated and to A.Niemi for the hospitality in the Uppsala University where the main part of this work was done. This work was partually supported by ISF grant MGK000 and by the grants of Russian fund of fundamental research RFFI 93-02-03379 and RFFI 93-02-14365. The research of A.M. was also supported by NFR-grant No F-GF 06821-305 of the Swedish Natural Science Research Council.

Note added. When this paper was finished we have learnt about [34] where related topics are discussed.

References

[1] M.Fukuma, H.Kawai, R.Nakayama, Int.J.Mod.Phys. **A6** (1991) 1385

[2] R.Dijkgraaf, H.Verlinde, E.Verlinde, Nucl.Phys **B352** (1991) 59

[3] A.Gerasimov et.al. Nucl.Phys.**B357** (1991) 565

[4] S.Kharchev et.al. Nucl.Phys. **B380** (1992) 181

[5] N.Seiberg, E.Witten, Nucl.Phys. **B426** (1994) 19; **B431** (1994) 484

[6] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov, A.Morozov, Phys.Lett. **B355** (1995) 466

[7] D.Gross, W.Taylor, Nucl.Phys. **B400** (1993) 181

[8] N.Hitchin, Duke.Math.Jour. **54** (1987) 91

[9] V.Inozemtsev, Comm.Math.Phys. **121** (1989) 629

[10] S.Kharchev et. al. Nucl.Phys.**B366** (1991) 569

[11] S.Kharchev et. al., Nucl.Phys. **B397** (1993) 339

[12] A.Gorsky, N.Nekrasov, Nucl.Phys. **B414** (1994) 213; **B436** (1995) 582

[13] A.Gerasimov et. al., Mod.Phys.Lett. **A6** (1991) 3079

[14] I.Krichever, Func.Anal.Appl.**14** (1980) 282

[15] I.Krichever, A.Zabrodin, hepth-9505039

[16] E.Witten, Comm.Math.Phys. **117** (1988); Nucl.Phys. **B340** (1990) 281; hepth 9312104

[17] S.Cecotti, C.Vafa, Phys.Rev.Lett, **68** (1992) 903; Comm.Math.Phys. **158** (1993) 569

[18] B.Dubrovin, *Geometry of 2D topological field theories*, Preprint SISSA-89/94/FM

[19] T.Nakatsu, K.Takasaki, hepth/9509162

[20] E.Martinec, N.Warner, hepth/9509161

[21] R.Donagi, E.Witten, hepth/9510101

[22] T.Eguchi, C.Yang, hepth/9510183

[23] P.Argires, A.Faraggi, Phys.Rev.Lett., **73** (1995) 3931
A.Klemm et.al Phys.Lett., **B344** (1995) 169

[24] A.Gorsky, N.Nekrasov, hepth/9401021

[25] S.Ruijsenaars, Comm.Math.Phys. **110** (1987) 191; Comm.Math.Phys. **133** (1990) 217

[26] N.Nekrasov, hepth/9503157
B.Enriquez, V.Rubtsov, alg-geom/9503010
O.Sheinman, hepth/9510165
M.Olshanetsky, hepth/9510143

[27] S.Kharchev et.al., Int.J.Mod.Phys. **A10** (1995) 2015

[28] S.Kharchev et.al., Mod.Phys.Lett. **A8** (1993) 1047

[29] S.Kharchev, A.Marshakov, Int.J.Mod.Phys. **A10** (1995) 1219

[30] I.Krichever, Func.Anal.& Apps. **22** (1988) 37; Comm.Math.Phys. **143** (1991) 415; preprint LPTENS-92-18

[31] O.Babelon, D.Bernard, Phys.Lett. **B317** (1993) 363
S.Ruijsenaars, in "Integrable and super integrable systems" ed. B.Kupershmidt (1990) 165

[32] P.Etingof, A.Kirillov Jr, Duke Math J. **74** (1994) 585

[33] S.Novikov, Func.Anal.& Appl. **24** (1990) 43
I.Krichever, *On Heisenberg relations for the ordinary linear differential operators*, preprint ETH (1990)
G.Moore, Comm.Math.Phys. **133**(1990),261

[34] E.Martinec, hepth/9510204